# Theory and Applications of Dynamical Systems 

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## Chapter 1 Introduction

In December 1995, Prof. L. O. Chua visited National Chiao-Tung University to deliver a key-note speech at an international conference on neural networks. He lectured on the theory and applications of CNN and showed us a $10 \times 10$ cells CNN chips, [12], [13].

The lecture was very interesting and impressive. At the end of his talk, he discussed some interesting open problems for mathematicians. During the break, Prof. S. N. Chow introduced me to Leon. Shui-Nee was visiting Tsing-Hua University for one year and was an old friend of Leon and me. I told Leon that I was interested in his problems and asked him to explain them to me further. Next morning, he brought a piece of paper with the hotel's letterhead, descrbing three open problems. He took some time to describe the problems in great detail to ensure that I had fully understood them. I then began my research on the mathematical foundations of CNNs and a long friendship with Leon.

Leon wants to know about the human brain and try to imitate the brain functions to produce very powerful, universal CNN chips for various applications, such as image processing, patterns recognition and others, and especially in areas in which the digital computers are not so effective.

A typical two-dimensional CNN is of the form
(1.1) $\frac{d x_{i, j}}{d t}=-x_{i, j}+z+\sum_{|k|,|l| \leq 1} a_{k, l} f\left(x_{i+k, j+l}\right)+\sum_{|k|,|l| \leq 1} b_{k, l} f\left(u_{i+k, j+l}\right)$,

$$
(i, j) \in \mathbb{Z}^{2}, \text { and }
$$

$$
\begin{equation*}
x_{i, j}(0)=x_{i, j}^{0} . \tag{1.2}
\end{equation*}
$$

Where the output function $f$ is a piecewise linear function of the form
(1.3) $f(x)=\frac{1}{2}(|x+1|-|x-1|)$.


Fig. 1.1. Output function $y=f(x)$.
$A=\left(a_{k l}\right)$ is a feedback template, a spatial-invariant template, and $B=\left(b_{k l}\right)$ is a controlling template, $z$ is the biased term or threshold. The quantities $x_{i, j}$ denote the state at cell $c_{i j}$.

Figure 1.2 presents the CNN.


Fig. 1.2. CNN

Stationary solutions $\bar{x}=\left(\bar{x}_{i j}\right)$ of (1.1) are very important in studying CNNs, their outputs $\bar{y}=\left(f\left(\bar{x}_{i j}\right)\right)$ are called patterns. Chua and Yang [12] and later Lin and Shih [27] showed that (1.1) behaves like a gradient system when template $A$ is symmetric, meaning that, $a_{-k,-l}=a_{k, l}$ for all $|k|$ and $|l| \leq 1$. In this case, every trajectory tends to a stable stationary solution as time passes. For other templates, the trajectories could be periodic, quasi-periodic or chaotic [26], [33].

Among the stationary solutions, the mosaic solutions are stable and are crucial to studying the complexity of (1.1). A mosaic solution $\bar{x}$ satisfies $\left|\bar{x}_{i, j}\right| \geq 1$ for all $(i, j) \in \mathbb{Z}^{2}$. Its corresponding pattern $\bar{y}=$ $\left(f\left(\bar{x}_{i j}\right)\right)$ is called a mosaic pattern. In this case, $\left|f\left(\bar{x}_{i j}\right)\right|=1$.

Some basic problems in CNN theory can be stated as follows:
(I) Direct problem: Given any $P \subset P^{19}=\{(z, A, B): A$ and $B$ are $3 \times$ 3 real matrices and $z \in \mathbb{R}\}$, determine $\mathcal{M}(\mathcal{P})$, the set of all mosaic patterns of (1.1).
(II) Inverse Problem or Learning Problem: Given a set of stationary patterns $\mathcal{U}$, determine a set of parameters $\mathcal{P} \subset \mathcal{P}^{19}$, such that $\mathcal{U}=\mathcal{M}(\mathcal{P})$.
(III) Study the complexity of the set of mosaic patterns $\mathcal{M}(\mathcal{P})$ for each subset $\mathcal{P} \subset \mathcal{P}^{19}$.

Given $(z, A, B) \in \mathcal{P}^{19}$, the complexity of the set of mosaic patterns $\mathcal{M}(z, A, B)$ can be studied with reference to spatial entropy. Indeed, on finite lattice $\mathbb{Z}_{m \times n}$, the number of mosaic patterns on $\mathbb{Z}_{m \times n}$ is $\Gamma(m, n ; z, A, B)$. The spatial entropy of $\mathcal{M}(z, A, B)$ is defined by

$$
h(\mathcal{M}(z, A, B))=\lim _{m, n \rightarrow \infty} \frac{\log \Gamma(m, n ; z, A, B)}{m n} .
$$

The limit always exists, see [5], [8], [9], [10], [25].

Now, $\mathcal{M}(z, A, B)$ exhibits spatial chaos if $h(\mathcal{M}(z, A, B))>0$. In this case, $\Gamma(m, n ; z, A, B) \sim e^{h m n}$ as $m, n \rightarrow \infty . \mathcal{M}(z, A, B)$ describes pattern formation if $h(\mathcal{M}(z, A, B))=0$.

The following chapters presents some answers to those problems.

## Chapter 2 Local Patterns

This chapter investigates the generation of local patterns for CNN (1.1).

Without a controlling term, the stationary solution to (1.1) satisfies

$$
\begin{equation*}
-x_{i j}+z+\sum_{|k|,|l| \leq 1} a_{k, l} y_{i+k, j+l}=0, \text { for }(i, j) \in \mathbb{Z}^{2} \tag{2.1}
\end{equation*}
$$

The methods are illustrated initially by studying one-dimensional CNN with template $A=[r, a, s]$ : equation

$$
\begin{equation*}
\frac{d x_{i}}{d t}=-x_{i}+z+r y_{i-1}+a y_{i}+s y_{i+1}, \text { for } i \in \mathbb{Z}^{1} \tag{2.2}
\end{equation*}
$$

and the corresponding stationary equation

$$
\begin{equation*}
x_{i}=z+r y_{i-1}+a y_{i}+s y_{i+1}, \text { for } i \in \mathbb{Z}^{1} \tag{2.3}
\end{equation*}
$$

Firstly, consider the mosaic solution $x=\left(x_{i}\right)$ of Eq. (2.3).
If $x_{i} \geq 1$, i.e., $y_{i}=1$, then

$$
\begin{equation*}
(a-1)+z+r y_{i-1}+s y_{i+1} \geq 0 \tag{2.4}
\end{equation*}
$$

If $x_{i} \leq-1$, i.e., $y_{i}=-1$, then

$$
\begin{equation*}
(a-1)-z-\left(r y_{i-1}+s y_{i+1}\right) \geq 0 \tag{2.5}
\end{equation*}
$$

Equation (2.3) has four parameters $z, a, r, s$. The $(r, s)$-plane is initially partitioned as follows to solve Eqs. (2.4) and (2.5).


Fig. 2.1. Primary partition of $(r, s)$-plane.
However, in the ( $a, z$ )-plane, two sets of four straight-lines are important. The first set is

$$
\begin{equation*}
l_{k}^{+}: a-1+z+r y_{l}+s y_{r}=0 \tag{2.6}
\end{equation*}
$$

which is related to (2.4), and the second set is

$$
\begin{equation*}
l_{k}^{-}: a-1-z-r y_{l}-s y_{r}=0 \tag{2.7}
\end{equation*}
$$

which is related to (2.5), here $y_{l}$ and $y_{r} \in\{-1,1\}$ and $1 \leq k \leq 4$.

When $(r, s)$ lies in the open regions (I) $\sim(V I I I)$ the Figs. of (2.6) and (2.7) can be drawn like Figs. 2.2. and 2.3.


Fig. 2.2. Lines $l_{k}^{+}$.


Fig. 2.3. Lines $l_{k}^{-}$.

Combining Figs. 2.2 and 2.3, enables the ( $a, z$ )-plane to be partitioned as in Fig. 2.4.


Fig. 2.4. Partition of ( $a, z$ )-plane.
The symbols $[m, n]$ in Fig. 2.4 have the following meanings. Consider $(r, s)$ lies in region (I) in Fig. 2.1:

$$
\begin{equation*}
r>s>0 . \tag{2.8}
\end{equation*}
$$

Denote by

$$
\begin{equation*}
C_{1}^{+}=C_{4}^{-}=-r-s, C_{2}^{+}=C_{3}^{-}=-r+s, \tag{2.9}
\end{equation*}
$$

$$
C_{3}^{+}=C_{2}^{-}=r-s, C_{4}^{+}=C_{1}^{-}=r+s
$$

Then, $C_{4}^{+}>C_{3}^{+}>0>C_{2}^{+}>C_{1}^{+}$are the intersects of $l_{i}^{+}$and $l_{j}^{-}$on the $z$-axis as in Figs. 2.2 and 2.3, respectively.

With reference to the local patterns on 3 -cells, +1 is represented by the symbol + and -1 is represented by the symbol -.

Under the condition (2.8), the eight local patterns can be listed and ordered as follows.


Fig. 2.5. Ordering of local patterns in region (I).
Now, when $(a, z)$ lies in region $[m, n]$ in Fig. 2.4, the only admissible patterns are exactly, (1) $\cdots(1)$ and (1) $\cdots(1)^{\prime}$. For example, when (2.8) holds and $(a, z) \in[3,2]$, then only (1), (2), (3), (1) ${ }^{\prime}$ and (2) $)^{\prime}$ can be produced. This fact is equivalent to the holding of inequalities (2.4) and (2.5) if and only if $y_{i-1}, y_{i}, y_{i+1}$ are of the form (1), (2), (3), (1) and (2) ${ }^{\prime}$.

Similarly, in each region from (II) to (VIII), an ordering of eight local patterns on 3-cells can be defined as in Fig. 2.5. Fig. 2.6 presents the complete ordering diagrams. The order are arranged from the first to the fourth as the local patterns run from the bottom to the top. Therefore, for each region from (I) to (VIII), ( $a, z$ )-plane can be partitioned into $5 \times 5$ regions [ $m, n$ ], as in Fig. 2.4, $0 \leq m, n \leq 4$.

S


Fig. 2.6. Orderings of local patterns in $(r, s)$ plane.

The regions in $(a, z)$-plane are reduced when $(r, z)$ lies on the boundaries of open regions in $(r, s)$-plane. Indeed, when $r=s$, i.e., $A=$ $[r, a, r]$ is symmetric, then $C_{2}^{+}=C_{3}^{+}=0$. On the other hand, when $s=-r$, i.e., $A=[r, a,-r]$ is antisymmetric, then $r(+1)+s(+1)=$ $r(-1)+s(-1)=0$, as shown in Fig. 2.7. In these cases, region $[m, n]$ with $m=2$ or $n=2$ disappears and then the regions in the ( $a, z$ )-plane shrink to $4 \times 4$, as shown in Fig. 2.8.


Fig. 2.7. Orderings of local patterns when $s=r$ or $s=-r$.


Fig. 2.8. Partition of $(a, z)$-plane when $s=r$ or $s=-r$.
Finally, on the $r$-axis, where $s=0$ and the $s$-axis, where $r=0$. The local patterns are ordered as shown in Fig. 2.9.

-: either + or -

Fig. 2.9. Ordering of local patterns on $r$ and $s$-axis.
The regions [ $m, n$ ] in which $m$ or $n$ equals 1 or 3 disappear and the number of regions in the ( $a, z$ )-plane decreases further to $3 \times 3$, as shown in Fig. 2.10.


Fig. 2.10. Partition of $(a, z)$-plane when $s=0$ or $r=0$.
Combining the partitions of the $(r, s)$-plane and the $(a, z)$-plane, yields the following result for the Direct Problem.

Theorem 2.1 For one-dimensional CNN of Eq.(2.3). There are 200 open subregions $\mathcal{P}_{k}, 1 \leq k \leq 200$, of $\mathcal{P}^{4} \equiv\left\{(A, z): z \in \mathbb{R}^{1}\right.$ and $A=$ $[r, a, s]$ is a $1 \times 3$ real matrix $\}$ such that
(i) $\mathcal{P}^{4}=\bigcup_{k=1}^{200} \overline{\mathcal{P}_{k}}$,
(ii) $\mathcal{P}_{k} \bigcap \mathcal{P}_{l}=\varnothing$ if $k \neq l$,
(iii) $(A, z)$ and $(\tilde{A}, \tilde{z})$ in $\mathcal{P}_{k}$ for some $k$ if and only if they generate the same local patterns.

The method can be applied to Eq.(2.1), for the two-dimensional problem [18]. In this case, the parameters are $z$ and $A$ which is a $3 \times 3$ real matrix. The Direct Problem can be solved as follows.

Theorem 2.2. There is a positive integer $K$ and unique set of open subregions $\left\{\mathcal{P}_{k}\right\}_{k=1}^{K}$ of $\mathcal{P}^{10}=\left\{(z, A): z \in \mathbb{R}^{1}\right.$ and $A$ is a $3 \times 3$ real matrix.\} satisfying
(i) $\mathcal{P}^{10}=\bigcup_{k=1}^{K} \overline{\mathcal{P}_{k}}$,
(ii) $\mathcal{P}_{k} \bigcap \mathcal{P}_{l}=\varnothing$ if $k \neq l$,
(iii) $(A, z)$ and $(\widetilde{A}, \widetilde{z}) \in \mathcal{P}_{k}$ for some $k$ if and only if they generate the same local patterns.

The result can also be generated for a larger template $(2 d+1) \times$ $(2 d+1)$ matrix $A, d \geq 1$. For the details see [18].

## Chapter 3 One-dimensional CNN

The previous chapter described the generation of local patterns. This chapter discusses the extension of local patterns to the whole $\mathbb{Z}^{1}$. The two-dimensional case will be discussed in Part II "Pattern generation problems" below.

The method for extending to global patterns involves determining the related transition matrix for a given set of local patterns.

For simplicity, only $A=[r, a, s]$ is considered. The set $\sum_{3 \times 1}$ of all local patterns defined on 3 -cell can be arranged into the following ordering matrix $X_{3 \times 1}=\left[p_{i j}\right]_{4 \times 4}$.


## X: no admissible pattern

Fig. 3.1. Ordering matrix $X_{3 \times 1}$.

Notably, the symbols - and + are ordered by $\boxminus<\boxplus$ and then lexicographically for patterns on $\mathbb{Z}_{n \times 1}$, for $n \geq 2$.

Given a subset $\mathcal{B} \subset \sum_{3 \times 1}$, which is called a basic set, a transition matrix $T=T(B)$ can be introduced as follows, and $T=\left(t_{i j}\right)_{4 \times 4}$ is
defined as

$$
t_{i j}=\left\{\begin{array}{lll}
1 & \text { if } & p_{i j} \in \mathcal{B},  \tag{3.1}\\
0 & \text { if } & p_{i j} \notin \mathcal{B}
\end{array}\right.
$$

Example 3.1. On [3, 2] of (I), the basic set

$$
\mathcal{B}=\{---,--+,+++,++-,-++\} .
$$

Hence

$$
T=\left(\begin{array}{llll}
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1
\end{array}\right)
$$

Note that

$$
\begin{equation*}
\Gamma_{n+2} \equiv\left|T^{n}\right|=\text { the sum of all entries in } \mathbb{T}^{n}, \tag{3.2}
\end{equation*}
$$

and is the number of all admissible pattern on $\mathbb{Z}_{n+2}, n \geq 1$, which can be generated from $T$ (or $\mathcal{B}$ ). The spatial entropy is defined as

$$
\begin{equation*}
h(T)=\lim _{n \rightarrow \infty} \frac{\log \Gamma_{n}}{n} . \tag{3.3}
\end{equation*}
$$

The limit exists and equals $\log \rho(T)$, where $\rho(T)$ is the maximum eigenvalue of $T$ according to the Perron-Frobenius Theorem. When $h(T)>$ 0 , the spatial chaos occurs. When $h(T)=0$, the pattern formation arises. The following theorem indicates which regions in $\mathcal{P}^{4}$ have positive entropy.

Theorem 3.2. The open regions with positive entropy in $(a, z)$-planes for $(r, s)$ in (I) $\sim(\mathrm{VIII})$ are displayed in Fig.3.2, where $\lambda_{i}, 0 \leq i \leq 4$, is the largest root of the polynomial defined by

$$
\begin{array}{ll}
Q_{0}(\lambda)=\lambda-2, & \lambda_{0}=2, \\
Q_{1}(\lambda)=\lambda^{3}-\lambda^{2}-\lambda-1, & \lambda_{1} \doteq 1.839286, \\
Q_{2}(\lambda)=\lambda^{3}-2 \lambda^{2}+\lambda-1, & \lambda_{2} \doteq 1.754877, \\
Q_{3}(\lambda)=\lambda^{2}-\lambda-1, & \lambda_{3}=g \doteq 1.61803, \\
Q_{4}(\lambda)=\lambda^{3}-\lambda-1, & \lambda_{4} \doteq 1.324717 .
\end{array}
$$

The entropy in the region is $\log \lambda_{i}$.

$[4,4]$
[4,3]
$[4,2]$


(VII)
(V) (VI)

(VIII)

Fig. 3.2.

Remark 3.3. The results of Theorem 3.2 can be generalized to any template $A=\left[r_{1}, r_{2}, \ldots, r_{k}, a, s_{1}, s_{2}, \ldots, s_{k}\right], k \geq 1$.

Remark 3.4. Given a basic set $\mathcal{B} \subset \sum_{3 \times 1}$, the exact number $\Gamma_{B, n}$ of patterns with various boundary conditions-Dirichlet, Neumann and periodic on $\mathbb{Z}_{n \times 1}$ can be computed, [7].

## Chapter 4 Spatial disorder of inclined output function

In the previous chapter, the output function $f(x)$ is flat in the range $|x|>1$. This chapter considers the output function $f(x)$ which is nonflat in the range $|x|>1$. It produces some interesting new phenomena including spatial entropy [1], [2], [3], [4], [17].

The general three-piecewise linear output function is defined by

$$
f(x)=\left\{\begin{array}{cl}
m x+k-m & \text { if } x \geq 1,  \tag{4.1}\\
k x & \text { if }-1 \leq x \leq 1, \\
l x-k+l & \text { if } x \leq-1
\end{array}\right.
$$

where, $k>0$ and $l, m \geq 0$. Moreover, when $k=1$,

$$
f_{l, m}(x)=\left\{\begin{array}{cl}
m x+1-m & \text { if } x \geq 1,  \tag{4.2}\\
x & \text { if }-1 \leq x \leq 1, \\
l x-1+l & \text { if } x \leq-1
\end{array}\right.
$$

Furthermore, when $k=1$ and $l=m, f_{l, m}$ is symmetric with respect to the origin and is denoted by

$$
f_{m}(x)=\left\{\begin{array}{cl}
m x+1-m & \text { if } x \geq 1,  \tag{4.3}\\
x & \text { if }-1 \leq x \leq 1, \\
m x-1+m & \text { if } x \leq-1
\end{array}\right.
$$

$f_{m}$ is studied first. Consider $A=[r, a, s]$ and $z=0$ in the stationary equation (2.3):

$$
\begin{equation*}
x_{i}=r y_{i-1}+a y_{i}+s y_{i+1}, \tag{4.4}
\end{equation*}
$$

for $i \in \mathbb{Z}^{1}$. When $m>0$, the inverse function $g_{m}$ of $f_{m}$ exists and is given by

$$
g_{m}(v)=\left\{\begin{array}{cl}
\frac{1}{m} v-\frac{1}{m}+1 & \text { if } v \geq 1  \tag{4.5}\\
v & \text { if }-1 \leq v \leq 1 \\
\frac{1}{m} v-1+\frac{1}{m} & \text { if } v \leq-1
\end{array}\right.
$$

Using the inverse function $g_{m}$, Eq. (4.4) can be rewritten as

$$
\begin{equation*}
g_{m}\left(v_{i}\right)=r v_{i-1}+a v_{i}+s v_{i+1} . \tag{4.6}
\end{equation*}
$$

If $r=0$ and $s \neq 0$, then

$$
\begin{equation*}
v_{i+1}=\frac{1}{s}\left(g_{m}\left(v_{i}\right)-a v_{i}\right) \tag{4.7}
\end{equation*}
$$

Therefore, Eq. (4.7) describes trajectories of a one-dimensional iteration map $F_{m}$ defined by

$$
\begin{equation*}
F_{m}(v)=\frac{1}{s}\left(g_{m}(v)-a v\right) . \tag{4.8}
\end{equation*}
$$

However, if $r \neq 0$ and $s \neq 0$, let

$$
\begin{equation*}
u_{i+1}=v_{i}, \tag{4.9}
\end{equation*}
$$

then (4.6) can be rewritten as

$$
\begin{equation*}
v_{i+1}=\frac{1}{s}\left(g_{m}\left(v_{i}\right)-a v_{i}-r u_{i}\right) . \tag{4.10}
\end{equation*}
$$

Therefore, (4.9) and (4.10) are trajectories of two-dimensional iteration map

$$
\begin{equation*}
F_{m}(u, v)=\left(v, \frac{1}{s}\left(g_{m}(v)-a v-r u\right)\right) . \tag{4.11}
\end{equation*}
$$

This chapter focuses on the complexity of the one-dimensional map
$F_{m}$ in (4.8). For two-dimensional map (4.11), when $m$ is positive and sufficiently small, the Smale horseshoe appears, see [17]. For these maps, each bounded trajectory will corresponds to the outputs of bounded stationary solutions. Furthermore, if the maps are chaotic, then the stationary solutions of Eq. (4.4) are spatially disordered. However, only stable stationary solutions of to (4.4) should be considered. Therefore, the set of all stable bounded orbits $S_{m}$ of $F_{m}$ must be considered. If the entropy of $S_{m}$ is positive, then the stable stationary solutions to Eq. (4.4) represent spatial disorder, or spatial chaos.

The stability considered herein is linear stability. The following definitions are applied.

Definition 4.1. Let $\bar{x}=\left(\bar{x}_{i}\right)_{i=-\infty}^{i=\infty}$ be a stationary solutions to Eq. (4.4). Then $\bar{x}$ is called a non-transitional stationary solution if $\left|\bar{x}_{i}\right| \neq 1$ for all $i$. The linearized operator of these $\bar{x}$ is defined by

$$
\begin{equation*}
(\mathcal{L}(\bar{x}) \xi)_{i}=-\xi_{i}+a f_{m}^{\prime}\left(\bar{x}_{i}\right) \xi_{i}+s f_{m}^{\prime}\left(\bar{x}_{i+1}\right) \xi_{i+1} \tag{4.12}
\end{equation*}
$$

for $\xi=\left(\xi_{i}\right)_{i=-\infty}^{i=\infty} \in l^{2} . \bar{x}$ is called stable if all real parts of the eigenvalues of $\mathcal{L}$ are negative with eigenvector in $l^{2}$ and unstable otherwise.

Notably, since $f_{m}$ is not differentiable in transition state $x_{i}=1$, only non-transitional stationary solutions are considered.

We firstly state the following stability result; for the proof see [19].
Lemma 4.2. Assume $a>1, r=0$ and $s>0$ and $m>0$. Let $\bar{x}=\left(\bar{x}_{i}\right)_{i=-\infty}^{i=\infty}$ be a non-transitional stationary solution to Eq. (4.4). Then
(i) if there exists $i \in \mathbb{Z}^{1}$ such that $\left|\bar{x}_{i}\right|<1$, then $\bar{x}$ is unstable.
(ii) if $m(a+s)<1$ and $\left|\bar{x}_{i}\right|>1$ for all $i \in \mathbb{Z}^{1}$, then $\bar{x}$ is stable.

The stability requires only part of the graph of $u=F_{m}(v)$ which lie
in $\{(v, u):|v|>1$ and $|u|>1\}$ is relevant.
Thus, the map $F_{m}$ is considered is a gap map associated with the solid part of the graph in Fig. 4.1.


Fig. 4.1. Gap map $F_{m}$.

The main result for Eq.(4.4) with an inclined output function can be stated as follow.

Theorem 4.3. Suppose $z=0, r=0$ and $s>0$, and $a>s+1$. Denote

$$
\begin{align*}
& m_{\infty}=m_{\infty}(a, s)=\frac{a-s-1}{a(a-1)+s(a-2)}  \tag{4.13}\\
& m_{2}=m_{2}(a, s)=\frac{a-s-1}{a(a-1)+s(a-1)} \tag{4.14}
\end{align*}
$$

and $h(m)$ is the spatial entropy of $F_{m}$. Then there exists a strictly decreasing sequence $\left\{m_{p}\right\}_{p=2}^{\infty}$ with

$$
\lim _{p \rightarrow \infty} m_{p}=m_{\infty}
$$

such that
(i) $m \in\left[0, m_{\infty}\right), h(m)=\log 2$,
(ii) $m \in\left[m_{p}, m_{p-1}\right), h(m)=\log \lambda_{p}$, where $\lambda_{p}$ is the largest root of

$$
\begin{equation*}
\lambda^{2 p-2}-\left(\sum_{i=0}^{p-2} \lambda_{i}\right)^{2}=0 \tag{4.15}
\end{equation*}
$$

Moreover, $\lambda_{p}$ is strictly increasing in $p$ with $\lambda_{3}=g<\lambda_{p}<2$ for $p \geq 4$,
(iii) if $m_{2} \leq m<\frac{1}{a+s}$, then $h(m)=0$.

The entropy function $h(m)$ is a step function of $m$ of the form shown in Fig. 4.2. The result of Theorem 4.3 show that the entropy $h(m)$ is a devil-staircase like function and is decreasing in $m$.


Fig. 4.2. Entropy function $h(m)$.

## Proof of Theorem 4.3.

Denote the fixed points of $F_{m}$ by $A, O$ and $D$ with
$O=(0,0)$,

$$
\begin{aligned}
& A=\left(A_{1}, A_{2}\right)=\left(\frac{1-m}{1-m(a+s)}, \frac{1-m}{1-m(a+s)}\right), \\
& D=\left(D_{1}, D_{2}\right)=\left(\frac{m-1}{1-m(a+s)}, \frac{m-1}{1-m(a+s)}\right)=-A,
\end{aligned}
$$

and $B=\left(1, F_{m}(1)\right)$ and $C=\left(-1, F_{m}(-1)\right)$,
i.e.,

$$
B=\left(B_{1}, B_{2}\right)=\left(1, \frac{1}{s}(1-a)\right), C=\left(C_{1}, C_{2}\right)=\left(-1, \frac{1}{s}(a-1)\right) .
$$

The theorem is proven by verifying that $m_{p}$ satisfies

$$
\begin{equation*}
F_{m_{p}}^{p-1}(a-1)=1 \tag{4.16}
\end{equation*}
$$

and $F_{m_{p}}(v)$ has $2 p$ periodic cycle $\left\{F_{m_{p}}^{i}(c)\right\}_{i=0}^{2 p-1}$, for $p \geq 2$.
For simplicity, the proof for $p=2$ and $p=3$ are sketched, for $p \geq 4$, see [19].

When $p=2$, then $m_{2}$ is given by

$$
\begin{equation*}
\frac{a-2}{a(a-1)} \tag{4.17}
\end{equation*}
$$

and $R_{1}^{+}\left(m_{2}\right)=a-1$. Then, there is 4 -periodic cycle staring from $C$ following by $R^{+}, B, L^{-}$and back to $C$, see Fig. 4.3. The four stable subintervals $I_{j}, \quad 1 \leq j \leq 4$, have the following covering relation

$$
\begin{aligned}
& I_{1} \rightarrow I_{2}, \\
& I_{4} \rightarrow I_{3}, \\
& I_{2} \rightarrow I_{3},
\end{aligned}
$$

and

$$
I_{3} \rightarrow I_{2}
$$

The transition matrix $M(2)$ of the stable subintervals is given by

$$
M(2)=\left[\begin{array}{llll}
0 & 1 & 0 & 0  \tag{4.18}\\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right]
$$

Hence, the entropy $h\left(m_{2}\right)=0$.


Fig. 4.3. Graph $F_{m_{2}}(v)$ and its stable subintervals.

For $p=3$, the graph and $F_{m_{3}}(v)$ and its subintervals are given in Fig. 4.4. The covering relation for $2 p$ stable subintervals are given by

$$
\left\{\begin{array}{l}
I_{1} \rightarrow I_{2} \rightarrow \cdots \rightarrow I_{p-1} \rightarrow I_{p}  \tag{4.19}\\
I_{2 p} \rightarrow I_{2 p-1} \rightarrow \cdots \rightarrow I_{p+2} \rightarrow I_{p+1}, \\
I_{p} \rightarrow I_{k} \quad \text { for } k=p+1 \text { to } 2 p-1, \\
I_{p+1} \rightarrow I_{k} \quad \text { for } k=2 \text { to } p .
\end{array}\right.
$$

In particular, for $p=3$

$$
\begin{aligned}
& I_{1} \rightarrow I_{2} \rightarrow I_{3}, \\
& I_{6} \rightarrow I_{5} \rightarrow I_{4}, \\
& I_{3} \rightarrow I_{4} \cup I_{5},
\end{aligned}
$$

and

$$
I_{4} \rightarrow I_{2} \cup I_{3} .
$$

Therefore, the transition matrix is

$$
M(3)=\left[\begin{array}{llllll}
0 & 1 & 0 & 0 & 0 & 0  \tag{4.20}\\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]_{6 \times 6}
$$

The entropy $h\left(m_{3}\right)=\log \lambda_{3}, \lambda_{3}$ satisfies

$$
\lambda^{4}-(\lambda+1)^{2}=0,
$$

as in (4.15).


Fig. 4.4. Graph $F_{m_{3}}(v)$ and its stable subintervals.

Remark 4.4.
(i) When the biased term $z \neq 0$, the $F_{m}$ is no longer symmetric with respect to the origin. However, results like those from Theorem 4.3 were obtained in [2].
(ii) When the output function is asymmetric, i.e., $f_{l, m}$ with $l \neq m$. The situation is much more complicated. For $z=0$, the entropy function $h(l, m)$ has already been studied in [1]. Now, the devil-staircase-like behavior is exhibited in both $l$ and $m$ directions.

## Chapter 5 Bifurcations and chaos in two-cell CNN with periodic inputs

This chapter studies the bifurcation and temporal chaos in two-cell CNN with periodic inputs.

Consider the following two-cell CNN with input:

$$
\left\{\begin{array}{l}
\dot{x_{1}}=-x_{1}+a y_{1}+s y_{2}+b u(t),  \tag{5.1}\\
\dot{x_{2}}=-x_{2}+r y_{1}+a y_{2},
\end{array}\right.
$$

where the feedback template $A=[r, a, s]$ satisfies

$$
\begin{equation*}
a>1, a-1<r \text { and } a-1<-s . \tag{5.2}
\end{equation*}
$$

It is easy to verify that condition (5.2) implies that there exists a limit cycle $\Lambda_{0}(A)$ to (5.1) when $b=0$.

The input function (or forcing function) is

$$
\begin{equation*}
u(t)=\sin \frac{2 \pi}{T} t \tag{5.3}
\end{equation*}
$$

with period $T>0$ and amplitude $b>0$.
The main theme in studying (5.1) is to find out appropriate inputs such that complicated attractors appear. Indeed, Zou and Nossek [33] discovered a ladyshoe type chaotic attractor when

$$
\begin{equation*}
A=[1.2,2,-1.2], T=4 \text { and } b \cong 4.08 \tag{5.4}
\end{equation*}
$$

In this chapter, we recover their results and study more general situation.

The programs for studying bifurcation and chaos are as follow.
(I) Take $b=0$ and study how the sustained limit cycles $\Lambda_{0}(A)$ vary with the template $A=[r, a, s]$. The existence and uniqueness of
limit cycles will be studied.
(II) Fix template $A=[r, a, s]$, find possible range of input periods $T$ such that (5.1) exhibit chaotic behavior for suitable $b>0$. In particular, try to find the relation between $T$ and $T_{0}(A)$ such that (5.1) have complex trajectories for some $b>0$.
(III) Fix $A$ and $T$ obtained in (I) and (II), try to identify critical numbers of $b$, say, $b_{0}^{*}<b_{1}^{*}<\ldots<b_{k}^{*}$, which represent various types of trajectories of (5.1) and may cause distinct bifurcations when $b_{j}^{*}$ is crossing.

In section 5.1, the existence and uniqueness problem of periodic cycle to Eq. (5.1) when $b=0$ is considered. In section 5.2 , the bifurcations precede chaos is discussed. The FFT is used to study periodic and quasi-periodic attractors. In section 5.3, the temporal chaos is investigated. The effects of input period $T$ can be studied by examining the asymptotic limit cycle $\Lambda_{\infty}(T, A)$ with period $T$. Then, the study focus on (i) effect of the input amplitude, (ii) effect of the input period and (iii) effect of the varying templates.

## § 5.1. Limit cycles

This section discusses the existence and uniqueness of limit cycle to Eq. (5.1) when $b=0$. Since the nonlinear output function $f$ is piecewise linear, the phase-plane $\mathbb{R}^{2}$ can be divided into nine regions which are the mosaic (saturated) region $\mathcal{M}_{j}$, the transitional (partial saturated) region $\mathcal{T}_{j}$, and the interior (not saturated) region $I, j=1,2,3,4$, as in Fig. 5.1.


Fig. 5.1.

It is easy to see that periodic orbit does not lie entirely in the interior region $I$. Therefore, periodic orbits have to intersect the exterior region $\mathcal{E}$, here

$$
\begin{equation*}
\mathcal{E}=\mathbb{R}^{2}-I=\overline{\bigcup_{k=1}^{4} \mathcal{T}_{k} \cup \mathcal{M}_{k}} \tag{5.5}
\end{equation*}
$$

A periodic orbit $\Lambda$ is called an exterior (periodic) cycle if $\Lambda \subseteq \mathcal{E}$, otherwise $\Lambda$ is called a non-exterior cycle, i.e., $\Lambda \bigcap I \neq \varnothing$.

We firstly present the following result.
Theorem 5.1. Assume (5.2) and $b=0$, then
(i) limit cycles exist,
(ii) no more than two limit cycles are present in the exterior region $\mathcal{E}$, and
(iii) if $1<a \leq 2$, then at most one exterior limit cycle exists.

The multiplicity of exterior limit cycle can be stated as follows.
Theorem 5.2. Assume (5.2) and $b=0$, define function

$$
\begin{align*}
& R(r, a, s)=\delta\left\{\left(-\delta+\frac{s}{a} \xi\right)\left(\eta-\xi \kappa^{q}\right)+\left(\eta-\frac{\gamma \delta \eta}{a \xi}\right)\left(\delta-\gamma \chi^{q}\right)\right.  \tag{5.6}\\
& \left.+\left(\frac{-r s}{a^{2}}+\frac{r \delta}{a \xi}\right)\left(\eta-\xi \kappa^{q}\right)\left(\delta-\gamma \chi^{q}\right)\right\},
\end{align*}
$$

where

$$
\left\{\begin{array}{l}
\xi=r-a+1, \eta=r+a-1,  \tag{5.7}\\
\gamma=1-s-a, \delta=a-s-a, \\
q=\frac{1}{a-1}, \quad \kappa=\frac{\xi}{\eta} \text { and } \chi=\frac{r}{\delta} .
\end{array}\right.
$$

Then,
(i) there is a unique exterior limit cycle if $R>0$, and
(ii) there is no exterior limit cycle if $R<0$.

When $A$ is antisymmetric, i.e., $-s=r, R(r, a, s)$ can be reduced to

$$
\begin{equation*}
R(r, a)=\left(a(a-1)+r^{2}-r\right)-\kappa^{q}\left(a(a-1)+r^{2}+r\right) . \tag{5.8}
\end{equation*}
$$

## Proof of Theorem 5.1.

(i) Under the assumptions (5.2), the origin $O=(0,0)$ can be easily verified to be the only steady state of (5.1), moreover, $O$ is an unstable spiral. Indeed, the associated eigenvalues at $O$ are given by

$$
\lambda_{ \pm}=a-1 \pm i \sqrt{-r s}
$$

By Poincaré-Bendixson Theorem, a limit cycle exists. Apart from the origins, all trajectories will tend to one of the limit cycles as $t \rightarrow \infty$.


Fig. 5.2. A typical orbit of (5.1) with initial condition ( $\alpha,-1$ ), and $1 \leq \alpha \leq a-s$.
(ii) Periodic solutions as in Thiran [32] are constructed to show that no more than two periodic orbits exist in exterior region $\mathcal{E}$.

Now starting at the point $(\alpha,-1)$ at $t=0$, where $1 \leq \alpha \leq a-s$, the trajectory $\Gamma_{\alpha}$ in $\mathcal{T}_{1}$ is followed; it intersects $x_{2}=1$ at the point $\left(\alpha_{1}, 1\right)$ on $t=t_{1}, 1<\alpha_{1}$, enters $\mathcal{M}_{1}$; then intersects $x_{1}=1$ at
$\left(1, \beta_{2}\right)$ on $t=t_{2}$, enters $\mathcal{T}_{2}$; then intersects $x_{1}=-1$ at the point $\left(-1, \beta_{3}\right)$ on $t=t_{3}$, and finally enters $\mathcal{M}_{2}$ and intersects $x_{2}=1$ at the point $\left(\alpha_{4}, 1\right)$ on $t=t_{4}$, i.e.,

$$
\begin{align*}
& \left(x_{1}(0), x_{2}(0)\right)=(\alpha,-1), \\
& \left(x_{1}\left(t_{1}\right), x_{2}\left(t_{1}\right)\right)=\left(\alpha_{1}, 1\right), \\
& \left(x_{1}\left(t_{2}\right), x_{2}\left(t_{2}\right)\right)=\left(1, \beta_{2}\right),  \tag{5.9}\\
& \left(x_{1}\left(t_{3}\right), x_{2}\left(t_{3}\right)\right)=\left(-1, \beta_{3}\right), \\
& \left.\left(x_{1}\left(t_{4}\right), x_{2}\left(t_{4}\right)\right)=\left(\alpha_{4}\right), 1\right),
\end{align*}
$$

see Fig. 5.2. Since $b=0$, (5.1) is an autonomous equation. The periodic orbit cannot intersect itself. Therefore, by (i), $\Gamma_{\alpha}$ is a periodic (closed) orbit if and only if

$$
\begin{equation*}
\alpha_{4}=-\alpha . \tag{5.10}
\end{equation*}
$$

$\alpha_{1}, \beta_{2}, \beta_{3}, \alpha_{4}$ and $t_{1}, t_{2}, t_{3}, t_{4}$ must be computed in terms of $\alpha$. The following expressions can be straight-forwardly obtained. The details are omitted here. It is easy to verify

$$
\begin{equation*}
\alpha_{1}=a+\frac{s(1-r)}{a}+\left(\alpha-a+\frac{s(r+1)}{a}\right)\left(\frac{\xi}{\eta}\right)^{q}, \tag{5.11}
\end{equation*}
$$

$$
\begin{equation*}
\beta_{2}=a+r-\frac{\eta \gamma}{\alpha_{1}-a-s}, \tag{5.12}
\end{equation*}
$$

$$
\begin{equation*}
\beta_{3}=a-\frac{r(s+1)}{a}+\left(\beta_{2}-a-\frac{r(1-s)}{a}\right)\left(\frac{\gamma}{\delta}\right)^{q}, \tag{5.13}
\end{equation*}
$$

$$
\begin{equation*}
\alpha_{4}=\frac{\delta \xi}{\beta_{3}+r-a}+s-a . \tag{5.14}
\end{equation*}
$$

$\alpha_{4}$ is written as a function of $\alpha$ to show that (5.10) has at most two solutions for $\alpha \in[1, a-s]$. Indeed, in the following, $k_{i}, i=$ $1, \cdots, 17$, are constants that depend on $a, r, s$, but are independent
of $\alpha$,

$$
\begin{aligned}
\alpha_{1} & =k_{1} \alpha+k_{2} \\
\beta_{2} & =\frac{k_{4}}{k_{1} \alpha+k_{3}}+k_{5} \\
\beta_{3} & =k_{6} \beta_{2}+k_{7} \\
\alpha_{4} & =\frac{k_{9}}{\beta_{3}+k_{8}}+k_{10}
\end{aligned}
$$

and

$$
\begin{equation*}
\alpha_{4}=\frac{k_{9} \alpha+k_{13}}{k_{11} \alpha+k_{12}}+k_{14} \tag{5.15}
\end{equation*}
$$

Substituting (5.15) into (5.10) yields, a quadratic equation for $\alpha$, i.e.,

$$
k_{15} \alpha^{2}+k_{16} \alpha+k_{17}=0
$$

Therefore, (5.10) has at most two solutions in $[1, a-s]$.
(iii) Note that

$$
\begin{align*}
& \frac{\partial}{\partial x_{1}}\left(-x_{1}+a y_{1}+s y_{2}\right)+\frac{\partial}{\partial x_{2}}\left(-x_{2}+r y_{1}+a y_{2}\right)  \tag{5.16}\\
= & \left\{\begin{array}{lll}
a-2 & \text { if } \quad\left(x_{1}, x_{2}\right) \in \mathcal{T}_{i}, \\
-2 & \text { if } & \left(x_{1}, x_{2}\right) \in \mathcal{M}_{i},
\end{array}\right.
\end{align*}
$$

$1 \leq i \leq 4$. The sign is nonpositive if $a \leq 2$. The Dulac criteria rule out the second closed orbit in $\mathcal{E}$. The proof is complete.

To prove Theorem 5.2, it sufficies to prove following theorem.

Theorem 5.3. Assume (5.2) and $b=0$. Let $\xi, \eta, \gamma, \delta$ and $q$ be
given by (5.7).
(i) There is a periodic cycle in the exterior region $\mathcal{E}$ if the following conditions are satisfied.

$$
\begin{aligned}
& \left(E_{1}\right) \quad a-1+\frac{s(1-r)}{a}+\left(1-a+\frac{s(r+1)}{a}\right)\left(\frac{\xi}{\eta}\right)^{q} \geq 0 \\
& \left(E_{2}\right) \quad a-1-\frac{r(1+s)}{a}+\left(1-a-\frac{r(1-s)}{a}\right)\left(\frac{\gamma}{\delta}\right)^{q} \geq 0
\end{aligned}
$$

In particular, if $A$ is antisymmetric, i.e., $-s=r,\left(E_{1}\right)$ and $\left(E_{2}\right)$ is equivalent to

$$
\text { (E) } \quad a(a-1)+r(r-1)-[a(a-1)+r(r+1)]\left(\frac{\xi}{\eta}\right)^{q} \geq 0
$$

(ii) There is no periodic orbit in the exterior region $\mathcal{E}$ if one of the following conditions holds.

$$
\left(N_{1}\right) \quad a-1+\frac{s(1-r)}{a}+\frac{s \xi}{a}\left(\frac{\xi}{\eta}\right)^{q}<0
$$

or

$$
\left(N_{2}\right) \quad a-1-\frac{r(1+s)}{a}-\frac{r \gamma}{a}\left(\frac{\gamma}{\delta}\right)^{q}<0
$$

In that case, all periodic cycles are necessary intersect the interior region $\mathcal{I}$.

## Proof.

The existence results are first proved.
It is easy to verify that if $\Lambda$ is an exterior periodic cycle then
$\Lambda \bigcap\left\{\left(x_{1},-1\right) \mid x_{1} \in[1, a-s]\right\} \neq \emptyset$.
From (5.7) and ( $E_{1}$ ),

$$
\alpha_{1}(\alpha) \geq 1 \quad \text { for all } \quad \alpha \in[1, a-s]
$$

Similarly, from (5.13) and ( $N_{1}$ ),

$$
\beta_{3}\left(\beta_{2}\right) \geq 1 \quad \text { for all } \beta_{2} \in[1, a+r] .
$$

Therefore, $x_{1}(\alpha, 1)$ maps $[1, a-s]$ into $[-a+s,-1]$. It implies $-x_{1}(\alpha, 1)$ maps $[1, a-s]$ into itself and then has a fixed point in $[1, a-s]$. Hence, (5.10) has at least one solution in $[1, a-s]$. This proves that exterior periodic cycle exists.

Clearly, $\left(E_{1}\right)$ and $\left(E_{2}\right)$ is equivalent to $(\mathrm{E})$ when $s=-r$.
Finally, from (5.12) and ( $N_{1}$ ),

$$
\alpha_{1}(\alpha)<1 \quad \text { for all } \quad \alpha \in[1, a-s],
$$

and from (5.13) and ( $N_{1}$ ),

$$
\beta_{3}\left(\beta_{2}\right)<1 \text { forall } \beta_{2} \in[1, a+r] .
$$

Hence, there is no exterior periodic cycle exists. The proof is complete.
Notably, $\left(N_{1}\right)$ and $\left(N_{2}\right)$ can be replaced by stronger conditions that can be verified easily as follows.

$$
0<a-1<r<1 \text { and }-s \geq \frac{a(a-1)}{1-r}
$$

and

$$
0<a-1<-s<1 \text { and } r \geq \frac{a(a-1)}{1+s}
$$

## § 5.2. Bifurcation Precede chaos

Intuitively, when the amplitude $b$ of input $b u(t)$ is small, period $T$ is different from the period $T_{0}$ of sustained limit cycle, then the (5.1) may have complicated trajectory like quasi-periodic but not chaotic.

Therefore, in the section, the impact of an input $b u(t)$ on its period $T$ and amplitude $b$ are studied for the bifurcation phenomena before chaos occurs.

Consider (5.1) with initial conditions

$$
\begin{equation*}
x_{1}(0)=\xi_{1} \text { and } x_{2}(0)=\xi_{2} . \tag{5.17}
\end{equation*}
$$

The solution of (5.1) and (5.17) is denoted by

$$
\left(x_{1}\left(t, \xi_{1}, \xi_{2}\right), x_{2}\left(t, \xi_{1}, \xi_{2}\right)\right)
$$

where $x_{i}\left(t, \xi_{1}, \xi_{2}\right) \equiv x_{i}\left(t, \xi_{1}, \xi_{2} ; b, T, A\right), i=1$ and 2 .
The $\omega$-limit set of (5.1) is denoted by

$$
\left(\omega\left(\xi_{1}, \xi_{2} ; b, T, A\right)\right)
$$

and the nonwandering set of (5.1) is denoted by

$$
\begin{equation*}
\Omega(b, T, A)=\overline{\cup \omega\left(\xi_{1}, \xi_{2} ; b, T, A\right)},\left(\xi_{1}, \xi_{2}\right) \in \mathbb{R}^{2} . \tag{5.18}
\end{equation*}
$$

Since the input is $T$-periodic, for a fixed parameter $A, T$ and $b, a$ two-dimensional Poincaré map of (5.1) can be defined as

$$
\begin{equation*}
F\left(\xi_{1}, \xi_{2}\right)=\left(x_{1}\left(T, \xi_{1}, \xi_{2}\right), x_{2}\left(T, \xi_{1}, \xi_{2}\right)\right) . \tag{5.19}
\end{equation*}
$$

Now, the study of the bifurcations problem of (5.1) is equivalent to the study of how $\Omega(b, T, A)$ changes when $b, T$ and $A$ vary.

In general, it is hard to identify $\Omega(b, T, A)$ and to know how it changes. Therefore, we concerned mainly with how "typical" trajectories vary with $b, T, A$. In our problem, a typical trajectory $\Gamma_{b} \equiv$ $\Gamma(b, T, A)$ and $\omega$-limit set $\omega_{b}=\omega(b, T, A)$ are chosen of (5.1) with the initial condition at the origin $O=(0,0)$. The $\omega$-limit set of Poincaré map is denoted by $\hat{\omega}(b, T, A)$.

To show $\Omega(b, T, A)$ is a chaotic attractor, the following conditions must be proven to hold
(i) $\Gamma(b, T, A)$ have a positive Lyapunov exponent,
(ii) $\hat{\omega}(b, T, A)$ is fractal,
(iii) FFT (Fast Fourier Transform) of $\Gamma(b, T, A)$ have a broad-band.

We first use FFT to study (5.1). Let $\Lambda_{0}$ be the sustained limit cycle for $b=0$ and is obtained from Theorem 5.1. Apply FFT to the $x_{1^{-}}$ component of $\Gamma_{b}$, i.e., $x_{1}(t, 0,0), t>0$. Pick up the first $N$ frequencies of these data, i.e., let $\left\{a_{k} e^{i \omega k t}\right\}_{k=1}^{N}$ satisfy

$$
\begin{equation*}
\left|a_{1}\right| \geq\left|a_{2}\right| \geq \ldots \geq\left|a_{N}\right| \geq\left|a_{\omega}\right| \tag{5.20}
\end{equation*}
$$

for other frequency $\omega$, where $a_{k}=a_{k}(b)$ and $\omega_{k}=\omega_{k}(b)$. Denote

$$
\begin{equation*}
\tau_{k}(b)=\frac{2 \pi}{\omega_{k}(b)} \tag{5.21}
\end{equation*}
$$

the period of the $k$ th mode. For simplicity, denote

$$
\begin{equation*}
T_{b}=\tau_{1}(b) \tag{5.22}
\end{equation*}
$$

which corresponds to the largest amplitude except for $T$-mode.
Let $a_{T}=a_{T}(b, T, A)$ be the amplitude of the period $T$ mode. The ratio

$$
\begin{equation*}
\mathcal{A}(b) \equiv \frac{\left|a_{T}(b)\right|}{\left|a_{1}(b)\right|} \tag{5.23}
\end{equation*}
$$

represents the relative strength of the $T$-mode with respect to the $T_{b^{-}}$ mode as $b$ varies. Equation (5.1) is called $T_{b}$ dominant if $\mathcal{A}(b) \ll 1$, the $T_{b}$ and $T$ modes are comparable if $\mathcal{A}(b) \simeq 1$ but $T$ is dominant if $\mathcal{A}(b) \gg 1$.

It is not difficult to verify

$$
\begin{equation*}
\lim _{b \rightarrow 0^{+}} T_{b}=T_{0} \tag{5.24}
\end{equation*}
$$

The normalized curves

$$
\begin{equation*}
R_{k}(b)=\frac{\tau_{k}(b)}{T_{b}} \tag{5.25}
\end{equation*}
$$

of $\tau_{k}(b), 1 \leq k \leq N$, are very useful for finding periodic orbits. To be more specific, in the ZN-(Zou-Nossek) case, $R_{k}(b)$ with $1 \leq k \leq 20$ and $b \in[0,4]$ are as in Fig. 5.3.


Fig.5.3. FFT of the largest 20 modes for the ZN-case :

$$
A=[1.2,2,-1.2] \text { and } T=4
$$

(1) The amplitude of the $T=4$ mode (represented by a red thick line in Fig. 5.3) grows steadily as $b$ increases in $(0,3.826)$. It is comparable to $T_{b}$ when $b$ is close to 4 , near the onset of chaos.
(2) Curve number (2) decreases and curve number (3) increases and merges into $T_{b} / 2$ and giving rise to $4 T$ periodic cycles. The $4 T$ cycle will survive for quite a large range of parameters in $(0.43,0.66)$. Curves merging is very common and induces a period cycle.
(3) The $T_{b} / 3$ mode maintains the largest parameters in $(0,3.826)$ and
gives rise to a $3 T$ periodic cycle in $(1.2,3.826)$.
(4) The dotted regions and window regions (stepped regions) interweave with each other. Stepped regions represent periodic cycles and dotted regions represent quasi-periodic orbits.

In the ZN -case, when $b \geq 3.826$, the strength of the $T$-mode is comparable with or larger than the strength of the $T_{b}$-mode. In the following, a heuristic argument is used to derive relations among for $b$, $T$ and $T_{0}$ when $T_{b}$ and $T$ are comparable.

Let

$$
\begin{equation*}
\gamma(t)=\Lambda_{0}^{\prime}\left(t+t_{0}\right), \tag{5.26}
\end{equation*}
$$

$\Lambda_{0}(t)$ be the limit cycle of (5.1) with $b=0$. The first equation of (5.1) is modelled as

$$
\begin{equation*}
\frac{d x}{d t}=\gamma(t)+b u(t) . \tag{5.27}
\end{equation*}
$$

Now, $\gamma(t)$ is a periodic function with period $T_{0}$ and $u(t)$ is a period function with period $T$. The two time scales of the functions $\gamma$ and $u$ can be normalized to a single time scale $\tau \in[0,1]$ by setting $t=T_{0} \tau$ for $\gamma$ and $t=T \tau$ for $u$. Hence,

$$
\begin{equation*}
x(t)=x(0)+T_{0} \Gamma(\tau)+b T U(\tau), \quad \tau \in[0,1], \tag{5.28}
\end{equation*}
$$

where

$$
\Gamma(\tau)=\int_{0}^{T_{0} \tau} \gamma(s) d s \quad \text { and } \quad U(\tau)=\int_{0}^{T \tau} u(s) d s
$$

are normalized periodic functions with period one. From (5.28), $x(t)$ can very maximally if $T_{0}$ and $b T$ have the same order of magnitudes and an appropriate time shift $t_{0}$ occurs in (5.28). Therefore, for a fixed
template $A=[r, p, s]$, let $T_{0}=T_{0}(A)$ be the period of the sustained limit cycle $\Lambda_{0}(A)$ (without input), define
(5.29) $\quad b^{*}(T)=\frac{c_{0} T_{0}(A)}{T}$,
where $c_{0} \sim 1$ is a constant that depends on $A$ and $T$.
From our experiences, for a given $A$ and $T, c_{0}=1$ in (5.29) is a good guess for the position at which to start the search for interesting ranges of $b$. $c_{0}(A, T)$ may decrease as $T$ increases. In the ZN -case and many other templates, (5.29) worked very well. See Fig. 5.4.


Fig. 5.4. Critical trajectories of $b_{1}^{*}, b_{2}^{*}, b_{3}^{*}$ and $b^{*}$, when

$$
A=[1.2,2,-1.2] \text { and } T=4 .
$$

Let $b_{0}^{*}>0$ such that

$$
\begin{equation*}
\left|a_{1}(b, T, A)\right|>\left|a_{T}(b, T, A)\right| \tag{5.30}
\end{equation*}
$$

hold in $\left(0, b_{0}^{*}\right), b_{0}^{*}$ can be assumed to be the least upper bound of $\tilde{b}$ such that (5.30) holds in $(0, \tilde{b})$.

When a periodic window appears on the open interval $B \subset\left(0, b_{0}^{*}\right)$,
the curve $R_{k}(b)$ will be a horizontal lines, or approximately one, on $B$. Furthermore, on $B$

$$
\begin{equation*}
T_{b}=\frac{m}{n} T \tag{5.31}
\end{equation*}
$$

for some positive integers $m$ and $n$ and $(m, n)=1$, i.e., $m$ and $n$ are relative prime. Therefore,

$$
\begin{equation*}
B_{m, n}=\left\{b \in\left(0, b_{0}^{*}\right) \left\lvert\, T_{b}=\frac{m}{n} T\right.\right\} \tag{5.32}
\end{equation*}
$$

is defined. Denote by

$$
\begin{equation*}
\left[T_{0} / T\right]=m^{*}, \tag{5.33}
\end{equation*}
$$

where $[x]$ is the largest integer which is equal to or smaller than $x$.
The solutions of (5.1) can be written explicitly on each of the nine regions $\mathcal{M}_{j}, \mathcal{T}_{j}$ and $\mathcal{I}$. Therefore, the exact solutions of periodic orbits in $B_{m, n}$ can be rigorously checked using a computer provided $n$ is not too large. The periodic cycles in $B_{m, n}$ are of period $m T$ and circle around the origin $O n$-times ( $n$-copies). This explanation partially prove of the following results.

Conjecture 5.4. Assume (5.30) holds.
Then
(i) $B_{m^{*}, 1} \neq \emptyset$, i.e., a stable $m^{*} T$ periodic cycle of (5.1)exists.
(ii) If (5.1) has another stable limit cycle with $m_{*} T$ period in $B_{m_{*}, n_{*}} \subset$ $\left(0, b_{0}^{*}\right)$ and $m_{*} / n_{*}<m^{*}$, then $\cup_{m_{*} / n_{*} \leq m / n \leq m^{*}} B_{m, n}$ is open and dense in ( $\hat{b}_{1}, \hat{b}_{2}$ ), where $\hat{b}_{1}=\inf B_{m^{*}, 1}$ and $\hat{b}_{2}=\sup B_{m_{*}, n_{*}}$, i.e., $T_{b}$ as a function of $b$ is a devil's staircase in $b$. See Fig. 5.5.


Figure 5.5. Devil's staircase like period function $T_{b}, A=[1.2,2,-1.2]$, (a) $T=4$ and (b) $T=2$.

(a) $b=0.5, T_{b}=4 T$

(b) $b=0.78, T_{b}=\frac{15}{4} T$


Fig. 5.6. Some typical orbits in $B_{m, n}$ prior to chaotic regions, $A=[1.2,2,-1.2]$ and $T=4$.


Figure 5.7. Some typical quasi-periodic orbit (a) and (c) and their $\omega$-limit set $\hat{\omega}$ of the Poincaré map (b) and (d), $A=[1.2,2,-1.2]$ and $T=4$.

## § 5.3. Chaos

This section discusses the chaotic phenomena that occurs when the strength of $\Gamma_{b}$ and input $b u$ are comparable. The study of asymptotic period cycles for various $T$ when $b$ is large is very useful.

Whether the $\omega$-limit sets $\omega_{b}$ and $-\omega_{b}$ can be separated from each other by the $x_{1}$-axis such that one lies in the upper half of phase-plane
and the other lies in the lower half of phase-plane is the main concern. The answer is affirmative when $T$ is relatively small. See Theorem 5.5. In any case, the system can always support a limiting cycle even for large $T$.

For a given template $A=[r, a, s]$,

$$
\begin{equation*}
w_{1}=\frac{x_{1}}{b} \tag{5.34}
\end{equation*}
$$

is written as $b \rightarrow \infty$, and the limiting equation for $w_{1}$ is

$$
\begin{equation*}
\frac{d w_{1}}{d t}=-w_{1}+u . \tag{5.35}
\end{equation*}
$$

The solutions of (5.34) are

$$
\begin{equation*}
w_{1}(t)=c e^{-t}+\frac{1}{1+\Omega^{2}}(\sin \Omega t-\Omega \cos \Omega t), \tag{5.36}
\end{equation*}
$$

where

$$
\Omega=\Omega(T)=\frac{2 \pi}{T}
$$

and $c$ is a constant. Consequently,

$$
\begin{equation*}
x_{1}\left(t, \xi_{1}, \xi_{2} ; b\right) \sim \frac{b}{1+\Omega^{2}}(\sin \Omega t-\Omega \cos \Omega t) \tag{5.37}
\end{equation*}
$$

for large $b$ and $t$. Notably,

$$
\begin{equation*}
-a-r<x_{2}\left(t, \xi_{1}, \xi_{2} ; b\right)<a+r \tag{5.38}
\end{equation*}
$$

always holds for large t . Now, (5.1) is assumed to have a asymptotic limit cycle $\Lambda_{\infty}$ with period $T$ as $b \rightarrow \infty$. From (5.38), $\Lambda_{\infty}$ will almost
be in the region $\left|x_{1}\right| \geq 1$. In the limit, denoted by $w_{2}(t)$ for $x_{2}(t ; b), w_{2}$ satisfies

$$
\begin{align*}
\frac{d w_{2}}{d t} & =-w_{2}+a+r \text { if } w_{2} \geq 1  \tag{5.39}\\
\frac{d w_{2}}{d t} & =(a-1) w_{2}+r \text { if } w_{2} \mid \leq 1 \\
\frac{d w_{2}}{d t} & =-w_{2}+r-a \text { if } w_{2} \leq-1
\end{align*}
$$

for a total time of $T / 2$ in the region of $w_{1} \geq 0$. Similar equations hold in region $w_{1} \leq 0$ for another $T / 2$ time. The separation theorem are stated as follows.

Theorem 5.5. The system (5.1) can support a limiting limit cycle $\Lambda_{\infty}$ with period $T$ provided
(i) in region $w_{2} \leq-1$ if

$$
\begin{equation*}
T<T_{1}^{*} \equiv 2 \log \frac{2 r}{r+1-a} \tag{5.42}
\end{equation*}
$$

(ii) in region $w_{2} \leq 0$ if

$$
\begin{equation*}
T<T_{0}^{*} \equiv 2\left(\log \frac{2 r}{r+1-a}+\frac{1}{a-1} \log \frac{r}{r+1-a}\right) \tag{5.43}
\end{equation*}
$$

Similarly, $-\Lambda_{\infty}$ lies in $w_{2} \geq 1$ and $w_{2} \geq 0$, respectively.


Fig. 5.8. Limiting limit cycle $\Lambda_{\infty}$.

## Proof.

(i) Assume that $\Lambda_{\infty}$ remains in the region $w_{2} \leq-1$ for $T / 2$ time. Then general solutions of (5.41) are

$$
\begin{equation*}
w_{2}(t)=c e^{-t}+r-a . \tag{5.44}
\end{equation*}
$$

For $1 \leq \beta \leq \alpha<a+r$, assume

$$
\begin{equation*}
w_{2}\left(t_{0}\right)=-\alpha, \text { and } w_{2}\left(t_{0}+\frac{T}{2}\right)=-\beta . \tag{5.45}
\end{equation*}
$$

Then (5.44) and (5.45) imply

$$
\begin{equation*}
T=2 \log \frac{\alpha+r-a}{\beta+r-a} . \tag{5.46}
\end{equation*}
$$

Since $1 \leq \beta \leq \alpha<a+r$, (5.42) follows.
(ii) Assume that $\Lambda_{\infty}$ remains in the region $w_{2} \leq 0$ for $T / 2$ time. Let

$$
\begin{align*}
& 0 \leq \beta \leq 1 \leq \alpha<a+r \\
& w_{2}\left(t_{0}\right)=-\alpha \\
& w_{2}\left(t_{0}+T^{\prime}\right)=-1  \tag{5.47}\\
& w_{2}\left(t_{0}+T / 2\right)=-\beta
\end{align*}
$$

where $0<T^{\prime}<T / 2$. Then from (5.40) and (5.41),

$$
\begin{equation*}
w_{2}(t)=c_{1} e^{-t}+r-a \quad \text { for } t \in\left[t_{0}, t_{0}+T^{\prime}\right], \tag{5.48}
\end{equation*}
$$

and

$$
\begin{equation*}
w_{2}(t)=c_{2} e^{(a-1) t}-\frac{r}{a-1} \quad \text { for } t \in\left[t_{0}+T^{\prime}, t_{0}+T / 2\right] . \tag{5.49}
\end{equation*}
$$

From (5.48),

$$
\begin{equation*}
T^{\prime}=\log \frac{\alpha+r-a}{1+r-a} . \tag{5.50}
\end{equation*}
$$

Similarly, from (5.47) and (5.49),

$$
\begin{equation*}
\frac{T}{2}-T^{\prime}=\frac{1}{a-1} \log \frac{r-\beta(a-1)}{r+1-a} . \tag{5.51}
\end{equation*}
$$

From (5.50) and (5.51),

$$
T=2\left\{\log \frac{\alpha+r-a}{r+1-a}+\frac{1}{a-1} \log \frac{r-\beta(a-1)}{r+1-a}\right\}
$$

which implies (5.43).

The proof is complete.

## Remark 5.6.

(i) The perturbation method can be used to prove that there exists a limit cycle $\Lambda_{b}$ with period $T$ for large $b . \Lambda_{b}$ lies in the region according to Theorem 5.5. The details are omitted here. See Fig 5.8.
(ii) For large $T, \Lambda_{\infty}$ can be proven to exist with period $T$. Furthermore, $\Lambda_{\infty}$ spends most of $T / 2$ time near $a+r$. Therefore, for large $T$ and large $b, \Lambda_{b}$ is a symmetric $T$-periodic cycle like a rhombus, with two vertices are close to $(1, a+r)$ and ( $-1,-a-r$ ), respectively. See Fig 5.9.

Remark 5.7. When $T$ satisfies either (5.42) or (5.43), $\Lambda_{\infty}$ and $-\Lambda_{\infty}$ are separated. Then $\Lambda_{b}$ and $-\Lambda_{b}$ are also separated when $b$ is large. For examples, in the ZN -case i.e. $A=[1.2,2,-1.2]$,

$$
\begin{equation*}
T_{1}^{*}=4.9698 \tag{5.52}
\end{equation*}
$$

and

$$
\begin{equation*}
T_{0}^{*}=8.5533 . \tag{5.53}
\end{equation*}
$$

Note that $T=4<T_{1}^{*}$ has been used in Zou \& Nossek [33], when $b$ decreases to some critical number $b_{\infty}^{*}, \omega_{b}$ and $-\omega_{b}$ cross each other and cause crises; chaos may occur when $b$ decreases a little further. See Figs. 5.9.(a) $\sim(\mathrm{d})$. If $T$ is too large, chaos may not occur, for example, in the ZN-case $A=[1.2,2,-1.2]$ and $T=10$, in that case, $\omega_{b}$ is like rhombus. See Fig. 5.10.


Figure 5.9. Crises induced by $\omega_{b}$ and $-\omega_{b}$ when $A=[1.2,2,-1.2]$ and $T=4$.


Fig. 5.10. $\omega_{b}=-\omega_{b}$ when $A=[1.2,2,-1.2]$ and $T=10$.
Now, a specific model of ZN-case is studied first to elucidate the methods and the chaotic behavior. We first study the effects of input amplitude.

## $\S \S$ 5.3.1. Effects of the input amplitude

The ZN-case, with $A=[1.2,2,-1.2]$ and $T=4$, is first considered as a model to help to discuss the bifurcations of chaotic phenomena. The Lyapunov exponents were computed for a long $b>0$.


Fig. 5.11. Lyapunov exponents diagram for the ZN-case with $b$ in $(5.27,5.13), b_{1}^{*}=3.98, b_{2}^{*}=4.284, b_{3}^{*}=4.365$ and $b^{*}=4.2697$

In each $W_{k}$ and $0 \leq k \leq 8$, the basic periodic cycle - the periodic solutions with the smallest period - can be identified. These windows are first compared in terms of the following characteristics of the basic periodic cycle in each $W_{k}$.

- Range of parameters in window,
- Period in $T$ units,
- Symmetry: "s" for symmetric and "a" for asymmetric cycles,
- Dominating mode in FFT : $T_{b}$ for superiority of $T_{b}$ and $T$ for input,
- Type of attractor : "I" for type I, "II" for type II. See 5.12.



Fig. 5.12. (a) Type I: the orbit $\Gamma_{b}$ circles around all three points $C^{+}$, $O$ and $C^{-}$. (b) Type II: the orbit $\Gamma_{b}$ does not circle around all of three points $C^{+}, O$ and $C^{-}$.

| characteristics | $(1)$ | $(2)$ | $(3)$ | $(4)$ | $(5)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $W_{0}$ | $(3.956,3.96)$ | 7 | a | $T$ | I |
| $W_{1}$ | $(3.992,4.008)$ | 11 | s | $T_{b}$ | I |
| $W_{2}$ | $(4.052,4.068)$ | 6 | a | $T$ | I |
| $W_{3}$ | $(4.092,4.104)$ | 10 | a | $T$ | I |
| $W_{4}$ | $(4.124,4.168)$ | 4 | a | $T_{b}$ | I |
| $W_{5}$ | $(4.252,4.260)$ | 9 | s | $T$ | I |
| $W_{6}$ | $(4.368,4.412)$ | 8 | a | $T$ | I |
| $W_{7}$ | $(4.42,4.431)$ | 4 | a | $T$ | II |
| $W_{8}$ | $(4.433, \infty)$ | 2 | a | $T$ | I |

Fig. 5.13. Characteristics in different windows $W_{k}, 0 \leq k \leq 8$.



Fig. 5.14. Typical Poincaré section in chaotic regions and basic periodic cycles in windows, $A=[1.2,2,-1.2]$ and $T=4$.

Considering $W_{k}$ carefully, for example, in $W_{4}$, reveals that at the middle point of $(4.124,4.168)$ the $4 T$ basic periodic cycle is asymmetric. To its left, a sequence of periodic-doubling occur; to its right is a quasi-periodic region. See Fig. 5.15. Similarly, $W_{5}$ includes a sym-
metric $9 T$ basic periodic cycle, with periodic-doubling to its left and a quasi-periodic region to its right.


Fig. 5.15. Bifurcations diagram for ZN -case with $b$ in window

$$
W_{4}=(4.124,4.172)
$$

Then we study the impacts of the input periods.

## $\S \S$ 5.3.2. Impact of the input periods

In the search for chaotic regions, Theorem 5.5 is first applied $T \leq T_{0}^{*}$ as in (5.43). For those $T, b$ near

$$
\begin{equation*}
b^{*}(T)=\frac{c_{0} T_{0}(A)}{T} \tag{5.54}
\end{equation*}
$$

is first tried. Computing three critical trajectories at $b_{1}^{*}, b_{2}^{*}$ and $b_{3}^{*}$ is important.

Normally, $b_{1}^{*}, b_{2}^{*}$ and $b_{3}^{*}$ near $b^{*}(T)$ whenever they exist. In the ZNcase, the graphs of $b^{*}, b_{1}^{*}, b_{2}^{*}$ and $b_{3}^{*}$ are drawn and compared with the regions in which (5.1) has a positive Lyapunov exponent ( $\geq 0.02$ ). The chaotic regions of (5.1) are markered by o in Fig. 5.16.


Fig. 5.16. Critical numbers $b^{*}, b_{k}^{*}, k=1,2,3$ for varying $T$, $A=[1.2,2,-1.2]$.

## $\S \S$ 5.3.3. Varying Templates

For $A=[1.2,2,-1.2], T \in[3.5,4.5]$ chaotic phenomena similar to $T=4$ were observed. For $T \geq 5$, no chaotic regions are found. For
$T \in[2,3]$, chaotic regions exist but $\hat{\omega}_{b}$ are not a ladyshoe. See Fig. 5.17. (a) $\sim(h)$.

(a) $T=2$ and $b=8.88$.

(c) $T=3$ and $b=5.844$.

(e) $T=3.5$ and $b=4.94$.
(b) $T=2$ and $b=8.872$.

(d) $T=3$ and $b=5.828$.

(f) $T=3.5$ and $b=4.901$.

(g) $T=4.5$ and $b=3.86$.

(h) $T=4.5$ and $b=3.822$.

Figure 5.17. Chaotic attractors and basic periodic cycles for $A=[1.2,2,-1.2]$ with varying $T$.

Finally, the bifurcation and chaos are studied when templates vary. The role of template $A=[r, a, s]$ is fundamental. It governs the basic dynamics among the inputs.

Let $\lambda_{1}(b, T, r, p, s)$ be the largest Lyapunov exponent of $\omega(b, T, r, p, s)$ and define

$$
\begin{equation*}
\lambda_{1}^{*}(r, p, s)=\sup _{b>0, T>0} \lambda_{1}(b, T, r, p, s) . \tag{5.55}
\end{equation*}
$$

Instead of considering all $b>0$ and $T>0$ in (5.55), denote

$$
\begin{align*}
& \lambda^{*}(r, p, s)=\max \left\{\lambda_{1}(b, T, r, p, s) \mid\right.  \tag{5.56}\\
& \left.\delta T_{0}^{*} \leq T \leq T_{0}^{*} \quad \text { and } \quad \delta_{1} b^{*}(T) \leq b \leq \delta_{2} b^{*}(T)\right\},
\end{align*}
$$

where $T_{0}^{*}$ is defined in (5.43) and $b^{*}(T)$ is defined in (5.54), $\delta$ and $\delta_{1}$ are small positive numbers, for example $\delta=\delta_{1}=0.1$, and $\delta_{2}=2$. Numerical evidence suggests that $\lambda^{*}(r, a, s)$ closely approximates to $\lambda_{1}^{*}(r, a, s)$.

Antisymmetric $A$ is first considered, i.e., $s=-r$, and write

$$
\begin{equation*}
\lambda^{*}(r, a)=\lambda^{*}(r, a,-r) . \tag{5.57}
\end{equation*}
$$



Fig. 5.18. The maximum Lyapunov exponent function $\lambda^{*}(r, 2)$ for $A=[r, 2,-r]$.


Fig. 5.19. The maximum Lyapunov exponent for $\lambda^{*}(r, 2, s)$ maker $\odot$ for $\lambda^{*}>0$ and $\cdot$ for $\lambda^{*}<0$.


Fig. 5.20. Some typical chaotic attractors for general $A=[r, 2, s]$.

### 5.3.4. Numerical Methods

The numerical methods has been used are recalled as.
Trajectory Numerically, for a given set of parameters, a template $A=$ $[r, a, s]$ that satisfies (5.2), an amplitude $b$ and period $T$, the system of differential equations is solved in FORTRAN 90 by calling a subroutine, RKF45, using the RUNGE-KUTTA-FEHLBERG $(4,5)$ methods described in [14], with step size $=0.05$, absolute error
$1 \times 10^{-10}$ and relative error $1 \times 10^{-8}$.
$\Omega$-limit set Since the $\omega$-limit set $\omega(b, T, A)$ is of greatest concern, $2 \times$ $10^{6}$ steps are taken in the RKF45 integration. The first $1 \times 10^{6}$ steps were ignored, and the following numerical methods applied to the remaining data; the last $1 \times 10^{6}$ points were taken as the $\omega$-limit set $\omega(b, T, A)$.

Poincaré map The $\omega$-limit set $\hat{\omega}(b, T, A)$ of Poincará $T$-map is taken every $T /$ stepsize points from $\omega(b, T, A)$. The relative error of the Poincaré map can be easily computed. For example, in the ZNcase $T=4$ with a step size $0.05,80$ steps must be integrated for each point on the Poincarè map. Therefore, the relative error $1 \times 10^{-8} \times 80=8 \times 10^{-7}$ is obtained for each successive point of the Poincaré map.

Lyapunov exponent The Lyapunov exponents are obtained by averaging eigenvalues of $\operatorname{DF}\left(\xi_{1}, \xi_{2}\right)$ on each point in $\hat{\omega}_{b}$. Here, a convergent condition is imposed that the relative error is less than $1 \times 10^{-4}$. Moreover, the first $1 \times 10^{6}$ steps in the numerical integration are ignored to accelerate the convergence.

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